Riemann Surfaces and Complex Analysis Notes

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A Riemann surface is a manifold with a complex structure. We will make this concrete.

> ***Definition: Topological Manifold*** *A topological space (X, T) is a topological manifold iff there is an $n \in \mathbb{N}$ such that there is neighbourhood of every point $x \in X$ homeomorphic to $\mathbb{R}^n$.*

It is often also required that (X, T) be second countable and Hausdorff.

> ***Definition: Smooth / Complex Manifold*** *A topological manifold can be made into a smooth/complex manifold by giving it a smooth/holomorphic atlas. An atlas is a cover of X $\{U_\alpha\}_{\alpha\in A}$ and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that the transition functions*
>
> $$\phi_\alpha|_{U_\alpha\cap U_\beta} \circ \phi_\beta^{-1}|_{\phi_\beta(U_\alpha\cap U_\beta)}$$
>
> *are smooth/holomorphic for every $\alpha, \beta \in A$.*

A Riemann surface is a complex 1 manifold, i.e. the homoemorphisms are into $\mathbb{R}^2 \cong \mathbb{C}$.

There is another definition of a Riemann surface as a ringed space.

> ***Definition: Riemann Surface*** *A Riemann surface is a topological manifold of dimension 2, (X, T) with a sheaf of $\mathbb{C}$ algebras $\mathcal{O}$. In particular we require that $\mathcal{O}(U)$ to be a subalgebra of $C^0(U)$ such that for every $x \in X$ $\exists U \in T$ and a homeomorphism*
>
> $$\varphi : U \to \Delta$$
>
> *such that the pullback of ϕ, denote it ϕ^* is an isomorphism of $\mathbb{C}$ algebras*
>
> $$\mathcal{O}(U) \xrightarrow{\sim} \mathcal{O}_\Delta(\Delta)$$
>
> $$\phi^*(f) = f \circ \phi$$

Definitions of Holomorphicity

Given an open subset $D \subseteq \mathbb{C}$ and a function $f : D \to \mathbb{C}$, we say that f is holomorphic iff one of the following equivilent conditions is met

- Identifying $\mathbb{C}$ with $\mathbb{R}^2$ (as a real vector space) then f becomes a function of the form $f(x, y) = (f_1(x, y), f_2(x, y))$. Then the holomorphic condition is that both $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$ exist and satisfy the Cauchy-Riemann equations

  $$\frac{\partial}{\partial x} f = i\frac{\partial}{\partial y} f$$

- Fixing the basis $\{1, i\}$ for $\mathbb{C}$ and identifying $\mathbb{C}$ with $\mathbb{R}^2$ we can then take $f : D \to \mathbb{R}^2$ if

  $$df : \mathbb{R}^2 \to \mathbb{R}^2$$

  exists it is just a matrix, then if df commutes with

  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  it is also holomorphic.

- the complex limit exists for all $z \in D$

  $$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

- $\exists M \in \mathbb{C}$ such that

  $$f(z+h) - f(z) = Mh - O(h^2)$$

- f is smooth and satisfies the Cauchy-Riemann equations

- Around every $x \in D$ there exists a neighbourhood and a Taylor series that converges to f on that neighbourhood

Recall also that for a bijective and holomorphic function the inverse is automatically holomorphic. Such a bijective holomorphic function is called biholomorphic.

Now we move on to thinking about maps between complex manifolds.

> ***Definition:*** *A map $f : S \to S'$ between complex manifolds is holomorphic iff for all the transition maps*
>
> $$\phi_\alpha^{-1} \circ f \circ \psi_\beta$$
>
> *are holomorphic.*

We say that two atlases on a complex manifold are equivilent when the identity between the topological space is holomorphic.

Properties of Holomorphicity

Given a holomorphic function between RS, picking charts makes it a holomorphic function between $\mathbb{C}$ and $\mathbb{C}$. The taylor expansion of this function will depend on the chart however two things are invariant under a change of coordinates

- The existence of the expansion
- The index of the first non-zero coefficient in the series i.e. the k value in

  $$\psi_\alpha \circ f \circ \varphi_\beta^{-1} = \sum_{i\geq k} a_i z^i$$

The value of this k is called the ***valuation*** of the holomorphic function, and we denote it $\nu_f(x) : S \to \mathbb{Z}$

Theorem. *If $f : X \to Y$ is a non-constant holomorphic function between RS then for all $x \in X$ there exist open sets $x \in U_x \subseteq X$ and $f(x) \in V_x \subseteq Y$ and charts $\varphi : U_x \to \mathbb{C}$, $\psi : V_x \to \mathbb{C}$ such that*

$$\psi \circ f \circ \varphi^{-1} = z^k$$

Holomorphic functions are locally powers. This has many useful corrolories

- Non-constant holomorphic functions take open sets to open sets.
- A holomorphic function with a local maximum at p has a neighbourhood on which it is constant (of p)
- If $f : X \to Y$ is holomorphic, non-constant, both X and Y are connected and X is compact then Y is compact and f is surjective.

- If $f : X \to \mathbb{C}$ is holomorphic and X is compact then f is constant.
- If $f : X \to Y$ is holomorphic, non-constant then the set
$$S = \{x \in X : \nu_f(x) > 1\}$$
has no point of accumulation in X.

Finally we recall the idea of a function being ***anti-holomorphic***, that is its conjugate is holomorphic.

- The composition of two antiholomorphic functions is holomorphic
- The composition of holomorphic and antiholomorphic is antiholomorphic

Riemann-Hurwitz

Given a holomorphic non-constant map between connected RS $f : X \to Y$ we call its ***degree*** the sum

$$\deg(f) = \sum_{x \in f^{-1}(s)} \nu_f(x)$$

where $s \in Y$ is arbitrary (the value is independent).

A polygonal decomposition of a surface X is a finite collection of points V and a finite collection of edges E, $\gamma_i : [0, 1] \to X$ such that they are homeomorphism from $(0, 1) \to \gamma(0, 1)$ and $\gamma(0), \gamma(1) \in V$. Then the connected components of $X \setminus \cup_i \gamma[0, 1]$ are called faces, F. The ***Euler characteristic*** of the surface is then defined as

$$\chi(X) = |F| - |E| + |V| = 2 - 2g$$

where g is the genus of the surface.

Theorem (Riemann-Hurwitz)**.** *If X and Y are connected compact RS and $f : X \to Y$ is a non-constant holomorphic function then*
$$\chi(X) = \deg(f)\chi(Y) - b$$
where b is called the ramification index and is a finite constant calculated by

$$b = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} (\nu_f(x) - 1)$$

Poles

A continuous map between topological spaces $p : Y \to X$ is said to be a ***covering*** of X if $\forall x \in X$ there is an open neighbourhood $x \in U \subseteq X$ such that $p^{-1}(U)$ is the disjoint union of open sets $\{V_i\}$ and $p|_{V_i}$ is a homeomorphism for every i.

The degree of a covering is the cardinality of the preimage of a point (with the technicalities swept under the rug).

If $p : Y \to X$ be non-constant holomorphic map between RS, a ***branched point*** of p is a point $y \in Y$ such that there is no neighbourhood V of y such that $p|_V$ is injective. p is further said to be a ***branched covering*** if there exists finite subsets $S \subseteq Y, T \subseteq X$ such that

$$f : Y \setminus S \to X \setminus T$$

is a covering.

Connect this to the valuation discussion

Meromorphic Functions

A meromorphic function on a RS X is a holomorphic map $X \to \mathbb{P}^1$, that is not constantly ∞.

Lemma. *A meromorphic function is locally of the form f_1/f_2 where f_1, f_2 are holomorphic functions (into $\mathbb{C}$) and f_2 is not constantly* 0*.*

A pole is a point that maps to ∞, its order is the valuation at that point. We denote the set of meromorphic functions on S by $\mathcal{M}(S)$.

Lemma.
$$\mathcal{M}(S) \cong \{(f, V) | f : V \to \mathbb{C} \text{ and } V = S \setminus F \text{ where } F \text{ is a discrete set}\}/\sim$$
where $(f, U) \sim (g, V)$ iff $f|_{U \cap V} = g|_{U \cap V}$. (isomorphic as fields)

Lemma. *If f and g are holomorphic, non-zero, on a compact and connected space, have the same set of zeroes and poles then $f = cg$ for some $c \in \mathbb{C}^\times$*

Theorem. *Fix a connected Riemann surface X. Then, there is a one-to-one correspondence between irreducible polynomials $P(T) \in \mathcal{M}(X)[T]$ and pairs (Y, f), where $\pi : Y \to X$ is a branched cover, and $f \in \mathcal{M}(Y)$, such that the pullback*
$$\pi^* : \mathcal{M}(X)[T] \longrightarrow \mathcal{M}(Y)[T],$$
has the property that $(\pi^* P)(f) = 0$.

Examples of Riemann Surfaces

1.6.1 Sphere

We can get a topological space via the one point compactification of $\mathbb{C}$

$$\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$$

Then we have the following atlas

$$\{(\mathbb{C}, id), ((\mathbb{C} \cup \infty) \setminus \{0\}, z \mapsto \frac{1}{z})\}$$

Note that there are only two charts so to show that its an atlas we only need to check the one overlap. The two opens overlap on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and we have that

$$\frac{1}{-} \circ id^{-1} = \frac{1}{-}$$

as a function $\mathbb{C}^* \to \mathbb{C}^*$ and it is also holomorphic (because 0 is not in $\mathbb{C}^*$).

1.6.2 Torus

Consider a lattice defined by the basis $1, \tau$ where $\tau \in \mathbb{C}$ is not colinear with 1. Then there is an action $\mathbb{Z} + \tau\mathbb{Z} \curvearrowright \mathbb{C}$ via addition. Thus we can take the quotient

$$\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \cong T^2$$

Which is the torus. We can see that this is the torus by considering the gluing diagram for the torus and comparing it to a fundamental domain for the lattice.

This also induces a complex structure on T^2 by choosing preimages of opens under the projection map.

1.6.3 Graphs of Functions

We have "functions" on $\mathbb{C}$ such as $\omega(z) = \sqrt{z}$, which are really relations, given by sets such as

$$\{(z, w) \in \mathbb{C}^2 : w^2 = z\}$$

We want to have a notion of when these relations are holomorphic. If S was a Riemann surface then we would know what it means for a function $S \to \mathbb{C}$ to be holomorphic. But notice that by simply swapping the tuples around we get

$$S' = \{(w, z) \in \mathbb{C}^2 : w^2 = z\}$$

which is the graph of $f(w) = w^2$ and so we have a homeomorphism to $\mathbb{C}$ by projection off of the first variable, then we could check that they were holomorphic etc. So because S could be seen as the graph of a function, actually a functoin, we can give it charts.

So in general the question then becomes " when is a set the graph of a function". The implicit function theorem provides us with some sufficient conditions.

Theorem (Implicit / Inverse Function Theorem)**.** *If $F : \mathbb{C}^2 \to \mathbb{C}$ is a function and $S = \{(w, z) \in \mathbb{C}^2 : F(w, z) = 0\}$, and moreover $\frac{\partial F}{\partial w}(p) \neq 0$ then there exists a neighbourhood $p \in U_1 \times U_2 \subseteq \mathbb{C}^2$ and a holomorphic function $g : U_1 \to U_2$ such that*

$$S \cap U_1 \times U_2 = \Gamma_g = \{(z, g(z)) : z \in U_1\}$$

Symmetrically if $\frac{\partial F}{\partial z}(p) \neq 0$ then there exists a neighbourhood $p \in U_1 \times U_2 \subseteq \mathbb{C}^2$ and a holomorphic function $h : U_2 \to U_1$ such that

$$S \cap U_1 \times U_2 = \Gamma_h = \{(h(z), z) : z \in U_2\}$$

If both derivatives are non-zero then the supplied h, g are mutually inverse.

1.6.4 Classification Of Surfaces

All RS are orientable and all orientable and compact surfaces are either the sphere or the connected sum of tori.

Differential Forms

In the past we have thought of differential forms as sections of certain bundles, however here we will consider them as more "formal" objects. Given a smooth surface S with an atlas $(\mathcal{U}_\alpha, \phi_\alpha)_\alpha$ then

- A differential 0 form is a smooth function from $S \to \mathbb{C}$
- A differnetional 1 form is a collection of smooth functions indexed by the same set as the atlas that we call ω, where $\omega_\alpha = f_\alpha dx_\alpha + g_\alpha dy_\alpha$ for some smooth zero forms f_α, g_α, that satisfies the following condition on the overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$$\omega_\beta = f_\alpha(\frac{\partial x_\alpha}{\partial x_\beta}dx_\beta + \frac{\partial x_\alpha}{\partial y_\beta}dy_\beta) + g_\alpha(\frac{\partial y_\alpha}{\partial x_\beta}dx_\beta + \frac{\partial y_\alpha}{\partial y_\beta}dy_\beta)$$

- A smooth 2-form is again a collection of maps η where locally $\eta_\alpha = f_\alpha dx_\alpha \wedge dy_\alpha$ and f_α is a smooth 0-form.

We can define a sheaf of n forms by $\mathcal{E}^n$, sending an open set to the collection of n forms on it (when considered as a manifold itself). We have a differential map that takes us from n forms to $n+1$ forms. Because a RS is a surface we can only have up to two forms (this is from the proper definition as sections) so we can consider the differential as an indexed map

$$d^n : \mathcal{E}^n \to \mathcal{E}^{n+1}$$

such that

$$d^0(f)_\alpha = \frac{\partial f}{\partial x_\alpha}dx_\alpha + \frac{\partial f}{\partial y_\alpha}dy_\alpha$$

$$d^1(\omega)_\alpha = \left(\frac{\partial g_\alpha}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial y_\alpha}\right)dx_\alpha \wedge dy_\alpha$$

A one form $\omega_\alpha = f_\alpha dx_\alpha + g_\alpha dy_\alpha$ is called a (1,0) form if $\forall\alpha$ $g_\alpha = 0$ or a (0,1) form if $f_\alpha = 0$.

We can say more in the case of a RS because the transition functions must be holomorphic they also satisfy the Cauchy Riemann equations. Thus if $(\mathcal{U}_\alpha, \phi_\alpha = (z, \bar{z}))_\alpha$ is our atlas then we know that

$$\frac{\partial z_\alpha}{\partial \bar{z}_\beta} = \frac{\partial \bar{z}_\beta}{\partial z_\alpha} = 0 \qquad \forall\alpha, \beta$$

We may want to replace the adjective "differential" with something along the lines of continuous", "measurable" etc. This simply means that the coefficient functions f_α has the property. This is not the case for holomorphic however where a one form can be written in a chart as $\omega_\alpha = f_\alpha dz + g_\alpha d\bar{z}$

- A smooth one form is said to be a $(1,0)$-form if $g_\alpha = 0$ for all α
- A smooth one form is said to be a $(0,1)$-form if $f_\alpha = 0$ for all α
- A holomorphic one form is a $(1,0)$ form such that f_α is holomorphic
- An anti-holomorphic one form is a $(0,1)$ form such that g_α is anti-holomorphic

Dolbeaut Operators

Intricately related to the differentials above are the operators

$$\partial^1 : \mathcal{A}^{(1,0)} \to \mathcal{E}^2$$

$$\bar{\partial}^1 : \mathcal{A}^{(0,1)} \to \mathcal{E}^2$$

sending $f dz \mapsto \frac{\partial f}{\partial z} dz \wedge d\bar{z}$ and $f d\bar{z} \mapsto \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$ and

$$\partial^0 : \mathcal{E}^0 \to \mathcal{A}^{(1,0)}$$

$$\bar{\partial}^0 : \mathcal{E}^0 \to \mathcal{A}^{(0,1)}$$

sending $f \mapsto \frac{\partial f}{\partial z} dz$ and $f \mapsto \frac{\partial f}{\partial \bar{z}} d\bar{z}$.

The kernel of $\bar{\partial}$ is exactly Ω^0, that is holomorphic functions.

Poincare Lemmas

Lemma. *If Δ is the unit disc in $\mathbb{C}$ and $f \in C^\infty(\Delta)$ then there is some $g \in C^\infty(\Delta)$ such that $\bar{\partial} g = f$*

Lemma. *If $U \subseteq S$ an open, connected, simplly connected, contractable space then*

- *Every closed 1-form is exact*
- *Every 2-form is exact*

Lemma. *closed i-form iff locally exact i-form.*

Categories

Abelian Categories

3.2.1 Snake Lemma

Exact Sequences and Exact Functors

Resolutions

3.4.1 Horeshoe Lemma

Derived Functors

3.5.1 Acyclic

If $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is a functor between abelian categories then we say that an object $J \in \mathscr{A}$ is $\mathcal{F}$-acyclic if $R^i\mathcal{F}(J) = 0$ for all $i > 0$.

Theorem. *If $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is left exact then the right derived functor is equal to the homology of $\mathcal{F}$ applied to any $\mathcal{F}$-acyclic resolution.*

So even though the right derived functor is defined in terms of injective resolutions it is actually sufficient to find an acyclic resolution.

Sheaves

A ***presheaf*** of $\mathscr{C}$ objects on a topological space (or site) is a functor $\mathcal{T}^{\text{op}} \to \mathscr{C}$, where $\mathcal{T}$ is the category given by the topological space (objects opens, morphisms inclusions). More generally a presheaf is any contravariant functor.

A ***sheaf*** of $\mathscr{C}$ objects over a topological space (X, T) is the data of

- A function $\mathcal{O} : T \to ob(\mathscr{C})$
- For each inclusion of open sets $V \subseteq U$ a (contravariant) function

$$res_U^V : \mathcal{O}(U) \to \mathcal{O}(V)$$

satisfying the following conditions

- $res_U^U = id_{\mathcal{O}(U)}$
- For any triple of open sets $W \subseteq V \subseteq U$

$$res_V^W \circ res_U^V = res_U^W$$

- For every open cover $\{U_i \subseteq U\}_{i \in I}$ of every open set U, if $s, t \in \mathcal{O}(U)$ and $res_U^{U_i}(s) = res_U^{U_i}(t)$ for all $i \in I$ the $s = t$
- Again for every open cover $\{U_i \subseteq U\}_{i \in I}$ of every open set U given a family $\{s_i \in \mathcal{O}(U_i)\}_{i \in I}$ such that $\forall i, j \in I$

$$res_U^{U_i \cap U_j}(s_i) = res_U^{U_i \cap U_j}(s_j)$$

  then there is a $s \in \mathcal{O}(U)$ such that $res_U^{U_i}(s) = s_i$ for every $i \in I$.

The first two conditions tell us that the restriction is functorial. The second two say that everything agrees locally.

Remark. This definition can be refrased homologically or in the language of equalizers.

Example. *The topological space*

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

(with the subspace topology) that assigns to each open set, U, the $\mathbb{C}$-algebra of holomorphic functions $U \to \mathbb{C}$ which we denote $\mathcal{O}_\Delta$.

Morphisms of Sheaves

Sheaves of abelian groups (for instance) over a topological space form a category. The morphism in this category are (just) natural transformations.

A sheaf morphism is ***surjective*** if the induced maps on the stalks is surjective. It is ***injective*** if the induced map on the stalks is injective.

relate to the definitions of injective and surjective that I know...

Lemma. *If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that*

$$\mathcal{F}(U) \to \mathcal{G}(U)$$

is surjective for all U then the morphism is a surjective morphism of sheaves.

The converse is not true, for instance the Dolbeault map $\mathcal{A}^{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)}$

Not clear why this is not surjective on opens

Types of Sheaves

First if we have a (pre)sheaf $\mathcal{F}$ on T, then a ***sub-(pre)sheaf*** is another sheaf $\mathcal{G}$ on T and a natural transformation $\eta : \mathcal{G} \to \mathcal{F}$ such that for every open set in T $\eta_U : \mathcal{G}(U) \to \mathcal{F}(U)$ is injective.

A morphism of sheaves is called **surjective** if the induced maps on the stalks are surjective at every point. In this case we call the codomain a *quotient sheaf* of the domain.

Definition: *A sheaf $\mathcal{F}$ on X is called soft if for every open $U \subseteq X$ and every closed $K \subseteq U$ the natural map is surjective*

$$\mathcal{F}(U) \to \operatorname*{colim}_{K \subseteq V \subseteq U} \mathcal{F}(V)$$

- The sheaf of locally constant functions is not soft
- The sheaf of holomorphic functions is not soft
- The sheaf of smooth functinos is soft
- The sheaf of smooth one forms is soft

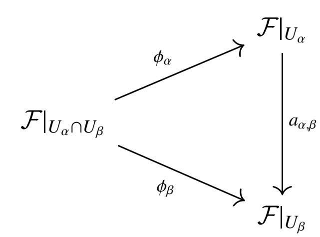
Lemma. *If $\mathcal{F}$ is a soft sheaf and $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is exact then for all U open $0 \to \mathcal{F}(U) \to \mathcal{F}'(U) \to \mathcal{F}''(U) \to 0$ is exact.*

Lemma. *If $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is exact and both $\mathcal{F}$ and $\mathcal{F}'$ are soft then so is $\mathcal{F}''$.*

Definition: *A sheaf $\mathcal{F}$ on X is said to be invertible if there exists an open cover $\{U_\alpha\}$ of X and an isomorphism of sheaves*

$$\mathcal{F}|_{U_\alpha} \xrightarrow{\phi_\alpha} \mathcal{O}_{U_\alpha}$$

which moreover satisfies the condition that when $U_\alpha \cap U_\beta \neq \emptyset$ there is a "transition function" $\exists a_{\alpha,\beta} \in \mathcal{O}_{U_\alpha \cap U_\beta}$ that is invertible and the diagram commutes



- $\mathcal{O}$ is invertible (structure sheaf)
- Ω^1 is invertible
- $\mathcal{O}_{\{-p\}} := \mathcal{I}_p$ is invertible (see the Riemann-Roch section)

Given an invertible sheaf $\mathcal{F}$ with isomorphisms ϕ_α and transition functions $a_{\alpha,\beta}$ then there is another sheaf $\mathcal{F}^{-1}$ with isomorphisms given by $\bar{\phi}_\alpha$ and transition functions $a_{\alpha,\beta}^{-1}$

Stalks

Given a sheaf $\mathcal{F}$ on X then the stalk of $\mathcal{F}$ at a point is

$$F_x = \operatorname*{colim}_{x \in U \text{ open}} F(U) \cong \{(f, U) : x \in U \text{ open }, f \in \mathcal{F}(U)\} / \sim$$

where $(f, u) \sim (g, v)$ when $f|_{U \cap V} = g|_{U \cap V}$. A single equivilence class $[(f, U)]$ is called the germ of f at x, and so the stalk is the set of all germs.

Stalks are colimits so for every x and U there is a natural map $\rho_x : \mathcal{F}(U) \to \mathcal{F}_x$; in fact this map and the stalk encode a lot of information

Lemma. $f = 0 \in \mathcal{F}(U)$ *if and only if* $\forall x \in U\ \rho_x(f) = 0$

Sometimes a sheaf will have the stronger property that if $\forall U$ open and connected and $\forall f \in \mathcal{F}(U)$ $f = 0$ iff $\exists x \in U$ such that $\rho_x(f) = 0$. Such a sheaf is said to satisfy the ***identiy theorem***.

Examples of Sheaves

The following are all sheaves

- The assignment of continous functinos to open sets on any topological space (C^0)
- The assignment of smooth n-forms on a smooth manifold ($\mathcal{E}^n$)
- The assignment of (anti)holomorphic n-forms to a RS ($\mathcal{A}^{(p,q)}$)
- The assigment of holomorphic functions on a RS ($\mathcal{O}$). This is a subsheaf of C^0 called the structure sheaf. This sheaf satisfies the identity theorem.

understand exactly why holomorphic functions satisfy the identity theorem but smooth functions do not

We can get more examples of sheaves by restricting already existing ones. If $\mathcal{F}$ is a sheaf on X and $U \subseteq X$ is open then we can define a new sheaf on U by

$$\mathcal{F}_U(V) = \mathcal{F}(U \cap V)$$

Homological Algebra of Sheaves

Exact Sequences

Homological algebra concepts can be applied to (pre)sheaves of Abelian groups. The category of presheaves abelian.

The cokernel of a sheaf (in the category of presheaves) is not necissarily a sheaf so the category of sheaves is not abelian.

Exercise. I f the sheaf of constant $\mathbb{Z}$ valued functions on a space is denoted $\underline{\mathbb{Z}}$ then there is an inclusion into the structure sheaf

$$\underline{\mathbb{Z}} \hookrightarrow \mathcal{O}$$

Show that the cokernel of this map is not a sheaf

Because the category of sheaves is not abelian we need another notion of exact sequences. We say that a chain of sheaves over X

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at the place $\mathcal{G}$ iff for every $x \in X$ we have

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact as a chain of abelian groups at $\mathcal{G}_x$.

Remark: There is a zero object for sheaves so we can talk about chains a priori (the maps square to 0), however not exactness.

Exact Functors

The inclusion Sh(X) $\hookrightarrow$ Presh(X) is a left exact functor.

If X is a topological space we have the functor of global sections on the category of sheaves

$$\Gamma : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{AbGrp}$$

which evaluates a sheaf on X. This is left exact and if we replace $Sh(X)$ with Presheaves on X we get an exact functor. The right derived functor of this is the sheaf cohomology.

The hom functor is left exact in both positions i.e. $Hom(a, -)$ and $Hom(-, a)$ is left exact.

In fact there is a natural isomorphism $\Gamma(X, -) \cong Hom_{\mathrm{Sh}(X)}(\mathbb{Z}, -)$ where $\mathbb{Z}$ is denoting the sheaf of constant integer valued functions on X.

Resolution Examples

Note that the category of (pre)sheaves has enough injectives.

Lemma. *Soft sheaves are* Γ*-acyclic.*

Lemma. *Any injective sheaf is soft.*

Structure Sheaf

$$0 \to \mathcal{O} \hookrightarrow \mathcal{A}^{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)} \to 0$$

is a soft resolution.

Constant Sheaf

$$0 \to \underline{\mathbb{C}} \to \mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \mathcal{E}^2 \to 0$$

is a soft resolution.

Holomorphic One Forms

$$0 \to \Omega^1 \to \mathcal{A}^{(1,0)} \xrightarrow{\bar{\partial}} \mathcal{A}^{(1,1)} \to 0$$

is a soft resolution.

General (Co)homology

A chain complex is a sequence of objects and morphisms in an (abelian) category such that the morphisms square to the zero morphism. It always makes sense to take the homology of a chain. Given a cochain complex, we can take its cohomology (the difference is only a matter of indexing, it is merely formal).

The homology of a chain $d_i : C_i \to C_{i-1}$ is given by

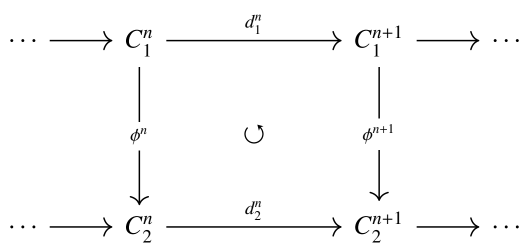
$$H_i = ker(d_i)/Im(d_{i+1})$$

The cohomology of a cochain $d_i : C_i \to C_{i+1}$ is

$$H_i = ker(d_i)/Im(d_{i-1})$$

To associate homology to a topological space then the real task is associating a (co)chain complex and then simply take its (co)homology.

This can be reformulated functorially, (co)homology is a functor $H^\bullet : Ch(\underline{A}) \to Ch(\underline{A})$ such that $H^\bullet(\phi)[c] = [\phi(c)]$. Recall that a chain map $\phi^\bullet : C_1^\bullet \to C_2^\bullet$ is a sequence of morphisms ($\phi^n : C_1^n \to C_2^n$) such that we get a commutative square at each place.



And homology maps such a chain map to

Cellular

To a Riemann surface with a given cellular decomposition we can associate the R-cellular **chain** complex

$$0 \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \to 0$$

Where C_i is the free R module on the i-cells of the decomposition.

Explicitly $C_2 = \text{Free}_R(\text{Faces})$, $C_1 = \text{Free}_R(\text{Edges})$ and $C_0 = \text{Free}_R(\text{Vertices})$. The maps are

$$\delta_2(face) = \text{boundary of face}$$

$$\delta_1(Edge) = \gamma(1) - \gamma(0)$$

If we know the chain complex above we can get the (R valued) cochain complex by homming into R i.e.

$$0 \to Hom(C_0, R) \xrightarrow{\delta_2^*} Hom(C_1, R) \xrightarrow{\delta_1^*} Hom(C_2, R) \to 0$$

notice that becuase Hom is contravariant the indicies now count down. Taking the cohomology of this is what we define to be the cellular cohomology.

De Rham

$\mathcal{E}^n$ is the sheaf of smooth n-forms on some RS. Recalling that the differential map d takes us from $\mathcal{E}^i$ to $\mathcal{E}^{i+1}$ we can form a cochain

$$0 \to \mathcal{E}^0(S) \xrightarrow{d^0} \mathcal{E}^1(S) \xrightarrow{d^1} \mathcal{E}^2(S) \to 0$$

From which we can take the cohomology. This is a chain of $\mathbb{C}$ algebras and so this is where the cohomology will land. We denote this $H^n_{dR}(X)$.

Remark: We consider these things to be $\mathbb{C}$ algebras because our sheaf $\mathcal{E}$ assigns the smooth n forms where the coefficients of our n forms are functions $U \to \mathbb{C}$. In differential topology for instance we would think of only functions $U \to \mathbb{R}$ and hence think of the De Rham cohomology as landing in $\mathbb{R}$ algebras.

6.3.1 Computing De Rham

Lemma.

$$H^0_{dR}(X) \cong \mathbb{C}^{\#\ connected\ components\ of\ X}$$

Dolbeaut

1 forms on a RS have a basis $dz, d\bar{z}$ so we can decompose $\mathcal{E} = A^{(1,0)} \oplus A^{(0,1)}$ where $A^{(1,0)}, A^{(0,1)}$ are now two new sheafs that assign one forms in the span of dz and $d\bar{z}$ respectively. This gives two cochains

$$0 \to A^{(1,0)}(S) \xrightarrow{\bar{\partial}} \mathcal{E}^2(S) \to 0$$

$$0 \to \mathcal{E}^0(S) \xrightarrow{\bar{\partial}} A^{(0,1)}(S) \to 0$$

The cohomology of these cochains is the $(1,0)$ and $(0,1)$ Dolbeaut cohomology respectively.

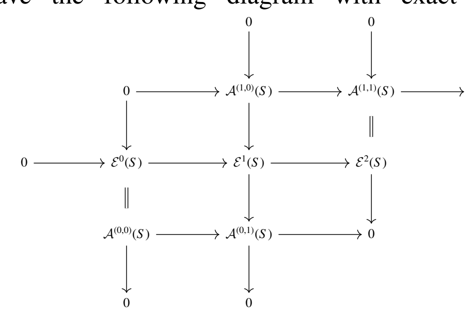
Remark: We are really using the (almost) complex structure to perform this decomposition because in general we are not able to assign charts in a consistent way such that z and $\bar{z}$ form a basis. This assignment is the almost complex structure.

6.4.1 Relation to the De Rham Complex

We can rewrite the De Rham complex as

$$0 \to \mathcal{E}^0(S) \xrightarrow{\partial + \bar{\partial}} A^{(1,0)} \oplus A^{(0,1)} \xrightarrow{\bar{\partial} - \partial} \mathcal{E}^2(S) \to 0$$

(which is the same as saying the center column splits below, using the inclusion map) Moreover we have the following diagram with exact columns



which by applying the snake lemma gives a long exact sequence in cohomology

$$0 \to H^0_{dR}(S) \to H^{(0,0)}(S) \to H^{(1,0)}(S) \to H^1_{dR}(S) \to H^{(0,1)}(S) \to H^{(1,1)}(S) \to H^2_{dR}(S) \to 0$$

Moreover

Theorem. *When S is compact RS we have*

$$H^i_{dR}(S) \cong \bigoplus_{p+q=i} H^{p,q}(S)$$

6.4.2 Computing Dolbeaut cohomology

Lemma.
$$H^{(0,0)}(X) = \mathcal{O}(X)$$
the space of holomorohic functions which if X is compact is $\mathbb{C}$

Lemma.
$$H^{(1,0)}(X) = \Omega^1(X)$$
which if X is compact and connected is $\mathbb{C}^g$ *where g is the genus of X.*

Lemma.
$$H^{(0,1)}(\Delta) = 0$$
where Δ *is the unit disk in* $\mathbb{C}$ *.*

Cech

In a sense this is the most general cohomology, as the others are special cases of it. Consider a sheaf $\mathcal{F}$ on a topological space X with a totally ordered cover $\{\mathcal{U}_\alpha\}_\alpha$ then the Cech complex is given by

$$0 \to \prod_\alpha \mathcal{F}(\mathcal{U}_\alpha) \to \prod_{\alpha<\beta} \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \cdots \to \prod_{\alpha_1<\cdots<\alpha_n} \mathcal{F}\left(\bigcap_i \mathcal{U}_{\alpha_i}\right) \to \cdots$$

The maps are discusting, alternating sums, fill in

The cohomology of this complex is denoted

$$\check{H}^i(\{\mathcal{U}_\alpha\}, \mathcal{F})$$

and is called the Cech cohomology of $\mathcal{F}$ with respect to the cover $\mathcal{U}_\alpha$

6.5.1 Computing Cech

Lemma.
$$\check{H}^0(\{\mathcal{U}_\alpha\}, \mathcal{F}) = \mathcal{F}(X)$$

6.5.2 Relation to Sheaf Cohomology

If the cover is Leray then the sheaf and Cech agree

> ***Definition:*** *A cover* $\{U_\alpha\}$ *of X is called a* $\mathcal{F}$*-Leray cover if for all* $i > 0, k \geq 1$ *(all possible intersections of all possible size)*
>
> $$H^i(U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}, \mathcal{F}) = 0$$

We say that a cover is $n-th$-Leray if only the cohomology for $0 < i \leq n$ vanishes. In this the $i-th$ sheaf cohomology still agrees with the $i-th$ Cech cohomology for $0 < i \leq n$.

Lemma. $\{U_\alpha\}$ *is* $\underline{\mathbb{C}}$*-Leray iff every intersction* $U_\alpha \cap U_\beta$ *is contractable and connected.*

Theorem (Cartan Theorem B)**.** *If U is a non-compact RS and* $\mathcal{F}$ *is an invertible sheaf then*

$$H^i(U, \mathcal{F}) = 0, \quad \forall i > 0$$

Sheaf

If X is a topological space ten recall we have the functor

$$\Gamma : Sh(X) \to AbGrp$$

$$\mathcal{F} \mapsto \mathcal{F}(X)$$

called the global sections functor. Because the category of sheaves always has enough injectives we can derive this functor, to get

$$H^i(X, -) \;:=\; R^i\Gamma Sh(X) \to AbGrp$$

So if $\mathcal{F}$ is a sheaf and we have an injective resolution

$$0 \to \mathcal{F} \to I^1 \to I^2 \to \cdots$$

then we apply the global sections functor to get the chain without $\mathcal{F}$

$$0 \to I^1(X) \to I^2(X) \to \cdots$$

and then take (co)homology giving the sheaf cohomology of $\mathcal{F}$.

6.6.1 Computing Sheaf Cohomology

Review the soft resolutions we gave above. These then tell us that

$$H^i(S, \mathcal{O}) = H^{(0,i)}(S)$$

$$H^i(S, \Omega^1) = H^{(1,i)}(S)$$

$$H^i(S, \underline{\mathbb{C}}) = H^i_{dR}(S)$$

Integrating Forms

One Forms

Let $\omega_\alpha = f_\alpha dx_\alpha + g_\alpha dy_\alpha \in \mathcal{E}^1(S)$ and γ be a piecewise smooth curve (for simplicity assume it lies inside a single chart, if not break it into pieces and sum the integrals) then we define

$$\int_\gamma \omega = \int_0^1 \left[f_\alpha(\gamma(t)) \frac{\partial x_\alpha}{\partial t} dt + g_\alpha(\gamma(t)) \frac{\partial y_\alpha}{\partial t} dt \right]$$

Then one can check that

- This is independent of chart
- This is independent of parametrization of γ
- If γ is closed then this integral depends only on ω up to an exact one form (immediate from Stokes)
- This integral depends only on γ up to a boundary (of a cellular decomposition say)

Because De Rham *cohomology* is the closed forms mod exact forms and cellular *homology* is the cycles mod boundaries the last two dot points show that in fact integration is defined "only up to homology" i.e. we have a well defined map

$$\int : H_1(S, \mathbb{Z}) \times H^1_{dR}(S) \to \mathbb{C}$$

Example. *Torus*

Two Forms

Give the smooth two form $\eta_\alpha = f_\alpha dx_\alpha \wedge dy_\alpha$ we define the integral over a compact connected set Ω with piece wise smooth boundary as

$$\int_\Omega \eta = \int_\Omega \eta_\alpha dx_\alpha \wedge dy_\alpha = \int \int \eta_\alpha dx_\alpha dy_\alpha$$

Theorem (Stokes)**.**

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega$$

This is essential in the proof that integration is defined up to homology above.

Closed Curves

A useful fact is that a closed curve on a surface is homotopic to a chain of edges, hence for integrals of closed forms the integrals of closed curves are $\mathbb{Z}$ linear combinations of the integral of the closed forms around one cells of the spcace (as these generate the edges).

Hodge Theory

Recall that we had a well difined map

$$\int : H_1(S, \mathbb{Z}) \times H^1_{dR}(S) \to \mathbb{C}$$

Now

Lemma. *ω a closed 1-form is exact iff for any closed curve C*

$$\int_C \omega = 0$$

this tells us that we may complexify

Our work in Hodge theory culminated in the Riemann-Hodge bilinear relations, from these we can prove

Lemma. *For a compact connected Riemann Surface S we have*

$$\dim H^0(S, \mathcal{O}) - \dim H^1(S, \mathcal{O}) = 1 - g$$

This is a special case of Riemann-Roch, which we now endeavor to prove.

Divisors

If S is a connected and compact RS then a ***divisor*** D is a formal $\mathbb{Z}$ linear combination of points of S

$$D = \sum_{s \in S} D_p \cdot p, \quad D_p \in \mathbb{Z}$$

such that $D_p = 0$ for all but finitely many p. Formally we may think of this as a function from $S \to \mathbb{Z}$ with a finite support.

We associate to a divisor a ***degree***

$$\deg D = \sum_{p \in S} D_p$$

Enter Sheaves

Given a divisor D we construct a sheaf associated

$$\mathcal{O}_D(U) = \left\{ f \in \mathcal{M}(U) : ord_p(f) \geq -D_p, \quad \forall p \in S \right\}$$

where

$$ord_p(f) = \begin{cases} 0, & \text{p is neither a pole or zero} \\ k, & \text{p is a zero order k} \\ -k, & \text{p is a pole order k} \end{cases}$$

Example. *There is another important sheaf*

$$\mathcal{I}_p(U) = \begin{cases} \mathcal{O}(U), & p \notin U \\ \{ f \in \mathcal{O}(U) : f(p) = 0 \}, & else \end{cases}$$

- *If $D = 0$ then $\mathcal{O}_D = \mathcal{O}$*
- *If $D = -p$ then $\mathcal{O}_D = \mathcal{I}_p$*
- *If $D = p$ then $\mathcal{O}_D = \mathcal{I}_p^{-1}$, where this is the inverse taken in the sense of $\mathcal{I}_p$ being an invertible sheaf.*

Definition: *The skyscraper sheaf at a point p is*

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C}, & p \in U \\ 0, & else \end{cases}$$

This sheaf is completely characterised by its stalks

$$(\mathbb{C}_p)_x = \begin{cases} \mathbb{C}, & x = p \\ 0, & else \end{cases}$$

We can now form a short exact sequence of sheaves

$$0 \to \mathcal{I}_p \to \mathcal{O} \to \mathbb{C}_p \to 0$$

The map $\mathcal{O} \to \mathbb{C}_p$ is the evaluation

$$f \mapsto \begin{cases} f(p), & p \in U \\ 0, & else \end{cases}$$

This can be shown to be exact using a long exact sequence in homology.

Riemann-Roch

The proof of this short exact sequence in essence shows that

Theorem. *If S is a compact connected RS and D is a divisor then*

$$\dim H^0(S, \mathcal{O}_D) - \dim H^1(S, \mathcal{O}_D) = 1 - g + \deg D$$

Abelian Differentials

Topic List

Week 1

- x Examples of $z = w^2$ and $w = \sqrt{(z^2-1)(z^2-2)}$
- x Riemann surface as manifold with complex structure
- x Definition and equivilent conditions for a function $\mathbb{C} \to \mathbb{C}$ to be holomorphic
- x Cauchy Riemann relation to commutation with linear operator
- x Definition of biholomorphic
- x Lemma that invertable holomorphic functions are all biholomorphic
- x Definition of holomorphic between two Riemann Surfaces
- x Definition of Riemann surface as a ringed space
- Problems with residue
- x Riemann Sphere
- x Torus (complex lattice)
- sqrt revisited.
- Inverse function theorem
- x Classification of surfaces

Week 2

- Valuation of a holomorphic functions
- holomorphic maps are locally powers
- Facts about holomorphic Functions
- degree of a map
- Polygonal decomposition of surface
- Riemann Hurewitz formula
- Covering space
- Branch points
- Branched coverings
- Meromorphic Functions

- Poles
- Isomorphism of fields between meromorphic functions
- zeroes
- Functions are determined by their zeroes and Poles
- Sqrt function again and that surfaces meromorphic functions
- Theorem 8.9 in For
- Return to residue
- smooth 0,1,2 forms

Week 3

- Classification of singularities
- holomorphic and antiholomorphic
- holomorphic functions are smooth functinos with zero antiholomorphic derivative
- differential map
- De Rham complex
- De Rham cohomology
- Global sections functor
- Sheaves generally
- Morphisms of sheaves
- sub sheaves
- Structure sheaf as subsheaf of continuous functions
- Stalk at a point
- Constant sheaf
- Identity theorem
- Stalks satisfy the identity theorem
- Surjective morphism of sheaves
- Complex of sheaves

Week 4

- 0th De Rham cohomology of a surface
- Dolbeaut cohomology
- Relation between De Rham and Dolbeaut
- Cellular chain
- Cellular homology
- Integrating one forms
- Exact and closed one forms
- Integration as a map on homology
- some stuff about sign
- missing lecture

Week 5

- Abelian categories
- Exact functor definition and examples
- Injective objects
- Resolutions
- Enough
- Horeshoe lemma
- Snake lemma
- Right derived functors
- Sheaf cohomology
- missing lecture
- soft sheaves
- Cech cohomology
- Cech resolution
- Leray cover
- Poincare lemma
- Cartans Thoerem B

Week 6

- Invertible sheaf
- Poincare lemma again
- Complexification
- Extension of integral to a map on complexified cohomology
- Closed forms and integrals around loops
- Star operator
- Harmonic function
- Holomorphic equivilent condition
- L2 space
- closed and coclosed
- Harmonic closed and coclosed decomp of L2 (Hodge decomp)
- Harmonic H (Weyls lemma)
- Every cohomology class (De Rham) has a unique harmonic representative
- Riemann Bilinear Relations

Week 7

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